Analyzing Bayesian Games

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Bayesian (Nash) Equilibrium

Bayesian (Nash) Equilibrium

- A plan of action for each player as a function of types that maximize each type's expected utility:

Bayesian (Nash) Equilibrium

- A plan of action for each player as a function of types that maximize each type's expected utility:
  - expecting over the actions of other players,

Bayesian (Nash) Equilibrium

- A plan of action for each player as a function of types that maximize each type's expected utility:
  - expecting over the actions of other players,
  - expecting over the types of other players.

Strategies

Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, strategies are defined as follows:

Strategies

Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, strategies are defined as follows:

- Pure strategy: $s_i : \Theta_i \mapsto A_i$
    - a choice of a pure action for player i as a function of his or her type.

Strategies

Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, strategies are defined as follows:

- Pure strategy: $s_i : \Theta_i \mapsto A_i$
  - a choice of a pure action for player i as a function of his or her type.
- Mixed strategy: $s_i : \Theta_i \mapsto \Pi(A_i)$
  - a choice of a mixed action for player i as a function of his or her type.

Strategies

Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, strategies are defined as follows:

- Pure strategy: $s_i : \Theta_i \mapsto A_i$
  - a choice of a pure action for player i as a function of his or her type.
- Mixed strategy: $s_i : \Theta_i \mapsto \Pi(A_i)$
  - a choice of a mixed action for player i as a function of his or her type.
- $s_i(a_i|\theta_i)$
  - the probability under mixed strategy s_i that agent i plays action a_i, given that i's type is θ_i.

Expected Utility

Three standard notions of expected utility:

Expected Utility

Three standard notions of expected utility:

- *ex-ante*
  - the agent knows nothing about *anyone*'s actual type;

Expected Utility

Three standard notions of expected utility:

- *ex-ante*
  - the agent knows nothing about *anyone's* actual type;
- *interim*
  - an agent knows her own type but not the types of the other agents;

Expected Utility

Three standard notions of expected utility:

- *ex-ante*
  - the agent knows nothing about *anyone's* actual type;
- *interim*
  - an agent knows her own type but not the types of the other agents;
- *ex-post*
  - the agent knows all agents' types.

Interim expected utility

- Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, i's *interim expected utility* with respect to type θ_i and a mixed strategy profile s is

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_i, \theta_{-i}).$$

Interim expected utility

- Given a Bayesian game (N, A, Θ, p, u) with finite sets of players, actions, and types, i's *interim expected utility* with respect to type θ_i and a mixed strategy profile s is

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_i, \theta_{-i}).$$

- i's *ex ante expected utility* with respect to a mixed strategy profile s is

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i).$$

Bayesian Equilibrium or Bayes-Nash equilibrium

A Bayesian equilibrium is a mixed strategy profile s that satisfies

$$s_i \in \arg\max_{s_i'} EU_i(s_i', s_{-i} | \theta_i)$$

for each i and $\theta_i \in \Theta_i$.

Bayesian Equilibrium or Bayes-Nash equilibrium

A Bayesian equilibrium is a mixed strategy profile s that satisfies

$$s_i \in \arg\max_{s_i'} EU_i(s_i', s_{-i}|\theta_i)$$

for each i and $\theta_i \in \Theta_i$.

The above is defined based on interim maximization. It is equivalent to an ex ante formulation:

Bayesian Equilibrium or Bayes-Nash equilibrium

A Bayesian equilibrium is a mixed strategy profile s that satisfies

$$s_i \in \arg\max_{s_i'} EU_i(s_i', s_{-i}|\theta_i)$$

for each i and $\theta_i \in \Theta_i$.

The above is defined based on interim maximization. It is equivalent to an ex ante formulation:

If $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, then this is equivalent to requiring that

$$s_i \in \arg\max_{s_i'} EU_i(s_i', s_{-i}) = \arg\max_{s_i'} \sum_{\theta_i} p(\theta_i) EU_i(s_i', s_{-i}|\theta_i)$$

for each i.

Bayesian (Nash) Equilibrium

- Explicitly models behavior in an uncertain environment

- Players choose strategies to maximize their payoffs in response to others accounting for:

  - strategic uncertainty about how others will play and

  - payoff uncertainty about the value to their actions.